MATH 147, FALL 2024: FINAL EXAM PRACTICE PROBLEMS

Below are problems to practice for the final exam. The problems below, together with the problems from the three midterm exams, are a good representation of what to expect on the final exam. There will also be a few short answer questions on the final exam.

1. Let $f(x,y) = \begin{cases} x^2+y^2, \text{ if } x^2+y^2<1 \\ 1, \text{ if } x^2+y^2 \geq 1. \end{cases}$. Determine at which points $f(x,y)$ is continuous.

Solution. Note that $g(x,y)=x^2+y^2$ and $h(x,y)=1$ are both continuous on all of $\mathbb{R}^2$. Thus, if we let D denote the unit disk $0 \leq x^2+y^2 \leq 1$, then $g(x,y)$ is continuous on the interior of D and $h(x,y)$ is continuous on $\mathbb{R}^2\backslash D$, and thus $f(x,y)$ is continuous on both the interior of D and $\mathbb{R}^2\backslash D$. For points (a,b) on the boundary of D, $a^2+b^2=1$, and we can consider $\lim_{(x,y)\to(a,b)} f(x,y)$. Let $\epsilon>0$. Since $g(x,y)$, as a function on $\mathbb{R}^2$, is continuous at (a,b) there exists $\delta>0$ such that $||(x,y)-(a,b)||<\delta$ implies $|g(x,y)-g(a,b)| = |g(x,y)-1|<\epsilon$. Taking the same δ, if $||(x,y)-(a,b)||<\delta$ and $(x,y)\in D$, then $g(x,y)=f(x,y)$, which gives $|f(x,y)-f(a,b)|=|g(x,y)-1|<\epsilon$. If $(x,y)\notin D$, then $f(x,y)-f(a,b)=1-1=0$, so $|f(x,y)-f(a,b)|<\epsilon$. Thus $f(x,y)$ is continuous at (a,b).

2. Show that the function $f(x,y) = \begin{cases} \frac{2^x-1)(\sin(y))}{xy}, \text{if } xy\neq 0 \\ \ln(2), \text{if } xy=0 \end{cases}$ is continuous at (0,0).

Solution. Set $g(x) = \begin{cases} \frac{2^x-1}{x} & \text{if } x\neq 0 \\ \ln(2) & \text{if } x=0 \end{cases}$ and $h(y) = \begin{cases} \frac{\sin(y)}{y} & \text{if } y\neq 0 \\ 1 & \text{if } y=0 \end{cases}$. Then $f(x,y)=g(x)h(y)$. Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = \{\lim_{x\to 0} g(x)\}\cdot\{\lim_{y\to 0} h(y)\}$. By L'Hospital's Rule, $\lim_{x\to 0} g(x)=\ln(2)$ and $\lim_{y\to 0} h(y)=1$. Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)=\ln(2)$.

3. Use the limit definition to show that $f(x,y)=5x+4y^2$ is differentiable at (2,1).

Solution. $\frac{\partial f}{\partial x}(2,1)=5$, $\frac{\partial f}{\partial y}(2,1)=8$, and $f(2,1)=14$, so $L(x,y)=5(x-2)+8(y-1)+14$. Therefore,

$$\begin{aligned} f(x,y)-L(x,y) &= 5x+4y^2-(5x-10+8y-8+14) \\ &= 4y^2-8y+4 \\ &= 4(y-1)^2. \end{aligned}$$

Thus, $\frac{f(x,y)-L(x,y)}{\sqrt{(x-2)^2+(y-1)^2}} = \frac{4(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}} \leq \frac{4(y-1)^2}{\sqrt{(y-1)^2}} = 4|y-1|$. Therefore,

$$\lim_{(x,y)\to(2,1)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-2)^2+(y-1)^2}} \leq \lim_{y\to 1} 4|y-1| = 0,$$

which shows that $f(x,y)$ is differentiable at (2,1).

4. From class, we saw that if the first order partial derivatives of $f(x,y)$ are continuous in a neighborhood of (a,b), then $f(x,y)$ is differentiable at (a,b). This problem shows why those conditions are necessary. Let

$$g(x,y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2}, \text{if } (x,y)\neq(0,0) \\ 0, \text{if } (x,y)=(0,0) \end{cases}.$$

Show that:

(i) $g(x,y)$ is continuous at $(0,0)$.
(ii) Use the limit definitions to show that $g_x(0,0)$ and $g_y(0,0)$ exist and are equal to 0.
(iii) Conclude that $L(x,y)=0$.
(iv) Show that $g(x,y)$ is not differentiable at (0,0).
(v) Show that $g_x(x,y)$ is not continuous at (0,0).

Solution. (i) Taking limits, we have.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} g(x,y) &= \lim_{r\to 0} \frac{2r^2\cos(\theta)\sin(\theta)(r\cos(\theta)+r\sin(\theta))}{r^2} \\ &= \lim_{r\to 0} r\cdot\{2\cos(\theta)\sin(\theta)(\cos(\theta)+\sin(\theta))\} \\ &= 0 \\ &= g(0,0),\end{aligned}$$

so $g(x,y)$ is continuous at (0,0).

(ii) $\frac{\partial g}{\partial x}(0,0) = \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$. $\frac{\partial g}{\partial y}(0,0) = \lim_{h\to 0} \frac{g(0,0+h)-g(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$.

(iii) $L(x,y) = 0(x-0) + 0(y-0) = 0$.

(iv) Thus, $g(x,y) - L(x,y) = g(x,y)$. If $g(x,y)$ were differentiable at (0,0), then the limit (when $(x,y) \neq (0,0)$)

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy(x+y)}{(x^2+y^2)^{\frac{3}{2}}}$$

should equal zero. If we take the limit along the line $y = x$, then $\frac{g(x,y)}{\sqrt{x^2+y^2}} = \frac{4x^3}{2\sqrt{2}x^3} = \sqrt{2}$, so the limit along the line $y = x$ as $x \to 0$ is not 0. Thus, $g(x,y)$ is not differentiable at (0,0).

(v) Differentiating the non-zero part of $g(x,y)$ gives $g_x(,y) = \frac{-2x^2y^2+4xy^3+2y^4}{(x^2+y^2)^2}$. If we take the limit along the line $y = 0$, then $\lim_{(x,y)\to(0,0)} g_x(x,y) = 0$, while if we take the limit along the line $x = 0$, the limit becomes 2. Thus, $\lim_{(x,y)\to(0,0)} g_x(x,y)$ does not exist, so that $g_x(x,y)$ is not continuous at (0,0).

5. Find and classify the critical points for $f(x,y) = x^4 - 4xy + 2y^2$.

Solution. To find critical points we solve

$$\begin{aligned} f_x &= 4x^3 - 4y = 0 \\ f_y &= -4x + 4y = 0. \end{aligned}$$

Form the second equation, we get $x = y$. Using this in the first equation gives $4x^3 - 4x = 0$, so that $x = 0, -1, 1$. Thus, the critical points are (0,0), (-1,-1), and (1,1). For the discriminant, we have

$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2)4 - (-4)^2 = 48x^2 - 16.$$

For (0,0): $D(0,0) = -16$, so that $f(x,y)$ has a saddle point at (0,0).

For (-1,-1): $D(-1,-1) = 32 > 0$ and $f_{xx}(-1,-1) = 12 > 0$. Thus, $f(x,y)$ has a relative minimum value (of -3) at (-1,-1).

For (1,1): $D(1,1) = 32 > 0$ and $f_{xx}(1,1) = 12 > 0$. Thus, $f(x,y)$ has a relative minimum value (of -3) at (1,1).

6. Find the absolute maximum and absolute minimum values of $f(x,y) = x^2y$ on the closed and bounded set $D : 0 \leq 4x^2 + 9y^2 \leq 36$.

Solution. Solving

$$\begin{aligned} f_x &= 2xy = 0 \\ f_y &= x^2 = 0 \end{aligned}$$

we see that $x = 0$, and y can be any real number. Thus, critical points in the interior of D are of the form $(0,y)$ with $0 \leq 9y^2 \leq 36$. However, $f(0,y) = 0$ for all such points. On the boundary of D, we must maximize and minimize $f(x,y)$ subject to the constraint $4x^2+9y^2=36$. Calling this equation $g(x,y)$, we set $\nabla f = \lambda \nabla g$ and solve the resulting system of equations

$$\begin{aligned} 2xy &= \lambda 8x \\ x^2 &= \lambda 18y \\ 4x^2+9y^2 &= 36. \end{aligned}$$

Notice that if x or y equal 0, then $f(x,y)=0$. So we can assume neither x nor y is zero. Dividing the first equation by $2x$ give $y=4\lambda$. Using this in the second equation gives $x^2=72\lambda^2$. Putting both of these into the constraint equations gives $4(72\lambda^2)+9(4\lambda)^2=36$, so that

$$3\lambda^2 = \frac{36}{144},$$

so that $\lambda = \pm\frac{1}{\sqrt{12}}$. Thus, $y=\pm\frac{4}{\sqrt{12}}$ and $x=\pm\sqrt{6}$. Thus, substituting these into x^2y gives $\pm\frac{24}{\sqrt{12}}=\pm 4\sqrt{3}$. Thus, on the domain D, the maximum value of $f(x,y)$ is $4\sqrt{3}$ and the minimum value is $-4\sqrt{3}$. Note that the critical points $(0,y)$ do note determine a minimum or maximum value of $f(x,y)$ on D.

7. Let S be the surface parametrized by $G(u,v)=(2u\sin(\frac{v}{2}), 2u\cos(\frac{v}{2}), 3v)$, with $0\le u\le 1$ and $0\le v\le 2\pi$.

(i) Find the tangent plane to S at the point $P=G(1,\frac{\pi}{3})$.
(ii) Find the surface area of S.

Solution.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 2\sin(v/2) & 2\cos(v/2) & 0 \\ u\cos(v/2) & -u\sin(v/2) & 3 \end{vmatrix} = (6\cos(v/2), -6\sin(v/2), -2u).$$

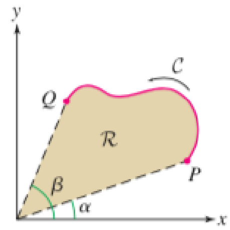
Therefore, $\mathbf{T}_u \times \mathbf{T}_v(1,\frac{\pi}{3}) = (3\sqrt{3},-3,-2)$, since $G(1,\frac{\pi}{3})=(1,\sqrt{3},\pi)$, for the tangent plane we have:

$$3\sqrt{3}(x-1)-3(y-\sqrt{3})-2(z-\pi)=0.$$

For the surface area, we have $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{36+4u^2} = 2\sqrt{9+u^2}$.

$$\begin{aligned} \text{Surface area} &= \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA \\ &= \int_0^{2\pi}\int_0^1 2\sqrt{9+u^2}\ dudv \\ &= 4\pi\int_0^1 \sqrt{9+u^2}\ du \\ &= 4\pi\{\frac{u}{2}\sqrt{9+u^2}+\frac{9}{2}\ln|u+\sqrt{9+u^2}|\}\Big|_0^1 \quad \text{(using a table of integrals)} \\ &= 4\pi\{\frac{\sqrt{10}}{2}+\frac{9}{2}\ln(1+\sqrt{10})-\frac{9}{2}\ln(3)\} \\ &= 4\pi\{\frac{\sqrt{10}}{2}+\frac{9}{2}\ln(\frac{1+\sqrt{10}}{3})\}. \end{aligned}$$

8. Let C be a curve from the point P to the point Q in the xy-plane. Let $\mathcal{R}$ be the region enclosed by C and the two radial lines from the origin to P and Q. (See the figure below.) Use Green's Theorem to show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ gives the area of $\mathcal{R}$, for $\mathbf{F} = -\frac{y}{2}\vec{i}+\frac{x}{2}\vec{j}$.



Solution. Let D denote the oriented closed curve forming the boundary of $\mathbb{R}$, namely, the line segment from the origin to P, followed by C, followed by the line segment from Q to the origin. We take $\mathbf{F} = (\frac{-1}{2}y, \frac{1}{2}x)$, so that by Green's Theorem, $\int_D \mathbf{F} \cdot d\mathbf{r}$ equals the area enclosed by the closed curve D. Let C_1 denote the line segment from (0,0) to $P = (a, b)$, C_2 denote the line segment from $Q = (c, d)$ to (0,0), and C the curve given in the illustration. Let $D = C_1 \cup C \cup C_2$ so that

$$\text{area}(R) = \int_D \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

We have to show $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Take $\mathbf{r}_1(t) = (at, bt)$, $0 \le t \le 1$ for the parametrization of C_1. Then $\mathbf{F}(\mathbf{r}(t)) = (-\frac{1}{2}bt, \frac{1}{2}at)$ and $\mathbf{r}'(t) = (a, b)$. Therefore $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\frac{1}{2}abt + \frac{1}{2}abt = 0$. So $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$. The calculation showing $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$ is similar.

9. Let C be the triangle with vertices (1,0,0), (0,2,0), (0,0,1). Compute $\int_C \mathbf{F} \cdot\ d\mathbf{r}$, for the vector field $\mathbf{F} = (x^2 + yz, x + y, y - z^2)$.

Solution. We have to integrate along each side of the triangle. $C_1 : \mathbf{r}(t) = (1 - t, 2t, 0).$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (-1, 2, 0)$. $\mathbf{F}(\mathbf{r}(t)) = ((1 - t)^2, 1 + t, 2t)$.

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 -(1-t)^2 + 2 + 2t\ dt \\
&= \int_0^1 -1 + 2t - t^2 + 2 + 2t\ dt \\
&= \int_0^1 1 + 4t - t^2\ dt \\
&= 1 + 2 - \frac{1}{3} = \frac{8}{3}.
\end{aligned}$$

$C_2 : \mathbf{r}(t) = (0, 2 - 2t, t)$, $0 \le t \le 1$, $\mathbf{r}'(t) = (0, -2, 1)$, $\mathbf{F}(\mathbf{r}(t)) = (2t - 2t^2, 2 - 2t, 2 - 2t - t^2)$.

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^1 -4 + 4t + 2 - 2t - t^2\ dt \\
&= \int_0^1 -2 + 2t - t^2\ dt \\
&= -2 + 1 - \frac{1}{3} = -\frac{4}{3}.
\end{aligned}$$

$C_3 : \mathbf{r}(t) = (t, 0, 1-t)$, $\mathbf{r}'(t) = (1, 0, -1)$, $\mathbf{F}(\mathbf{r}(t)) = (t^2, t, -(1-t)^2)$.

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 t^2 + (1-t)^2\ dt \\ &= \{\frac{t^3}{3} - \frac{(1-t)^3}{3}\}\Big|_0^1 \\ &= \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}.\end{aligned}$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{8}{3} - \frac{4}{3} + \frac{2}{3} = 2.$$

10. Let $f(x, y) = \sqrt{|xy|}$. Write out details showing:
   (a) $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist.
   (b) $f(x, y)$ is not differentiable at (0,0).
   (c) Part (b) does not contradict part (a).

Solution.

2.(a) $\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$. The calculation for $\frac{\partial f}{\partial y}(0, 0)$ is similar.

(b) From (a), the linear approximation to $f(x, y)$ at (0,0) is $L(x, y) = 0$. Therefore, $f(x, y) - L(x, y) = f(x, y)$. In order for $f(x, y)$ to be differentiable at (0,0), the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}}$$

should be 0. Taking the limit along the line $y = x$, we have

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{x\to 0} \frac{\sqrt{|x^2|}}{\sqrt{2}\sqrt{x^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore, $f(x, y)$ is not differentiable at (0,0).

(c) Part (b) does not contradict part (a) because the first order partial derivatives of $f(x, y)$ are not continuous at (0,0). In fact, the partial derivatives of $f(x, y)$ are not defined at all points (x, y) near (0,0), so we cannot evaluate the required limit to test continuity. To see this, let's try to calculate $\frac{\partial f}{\partial x}$ along the line $x = 0$, with $y \neq 0$.

$$\frac{\partial f}{\partial x}(0, y) = \lim_{h\to 0} \frac{f(0 + h, y) - f(0, y)}{h} = \lim_{h\to 0} \frac{|hy| - 0}{h} = |y| \lim_{h\to 0} \frac{|h|}{h},$$

which does not exist.

11. Evaluate $\int\int_S \text{Curl}\mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = (-y + z\sin(x), x, z^3)$ and S the surface defined by the equation $x^2 + \frac{y^2}{4} + z^2 + z^4x^2 = 1$, with $z \geq 0$.

Solution. We use Stoke's Theorem. The surface lies above the xy-plane, and intersects the xy-plane along the ellipse $x^2 + \frac{y^2}{4} = 1$. By Stokes Theorem, $\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$, for $C : (\cos(t), 2\sin(t), 0)$, with $0 \leq t \leq 2\pi$. Note that the orientation of C is consistent with what is required by Stoke's Theorem.

$\mathbf{r}'(t) = (-\sin(t), 2\cos(t), 0)$ and $\mathbf{F}(\mathbf{r}(t)) = (-2\sin(t), \cos(t), 0)$. Therefore,

$$\begin{aligned}\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^{2\pi} 2\ dt \\ &= 4\pi.\end{aligned}$$

12. Verify the Divergence Theorem for $\mathbf{F} = (-x^2, y^2, -z^2)$ and S rectangular box $[0,3] \times [-1,2] \times [1,2]$.

Solution. Div $(\mathbf{F}) = -2x + 2y - 2z$, therefore if B is the solid contained in the given rectangular box,

$$\begin{aligned}\int\int\int_B \text{Div } \mathbf{F}\ dV &= \int_1^2 \int_{-1}^2 \int_0^3 -2x + 2y - 2z\ dxdydz \\ &= \int_1^2 \int_{-1}^2 (-x^2 + 2xy - 2xz)\Big|_{x=0}^{x=3}\ dydz \\ &= \int_1^2 \int_{-1}^2 -9 + 6y - 6z\ dydz \\ &= \int_0^1 (-9y + 3y^2 - 6yz)\Big|_{y=-1}^{y=2}\ dz \\ &= \int_1^2 -18 - 18z\ dz \\ &= (-18z - 9z^2)\Big|_{z=1}^{z=2} \\ &= (-36 - 36) - (-18 - 9) \\ &= -45.\end{aligned}$$

To calculate the surface integral, we must sum the integrals over each face of the given rectangular box.

Front Face, S_1: S_1 is given by $(3, y, z)$, with $-1 \leq y \leq 2$, $1 \leq z \leq 2$ and $\mathbf{n} = i$. We will see the bounds on y and z are just used to calculate the area of the front face. The same will hold for the other five faces. So: $\mathbf{F}$ on S_1 is given by $(-9, y^2, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = -9$. Therefore,

$$\begin{aligned}\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_1} \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_1} -9\ dS \\ &= -9 \cdot \text{area}(S_1) \\ &= -9 \cdot 3 = -27.\end{aligned}$$

Back Face, S_2: S_1 is given by $(0, y, z)$, with $-1 \leq y \leq 2$, $1 \leq z \leq 2$ and $\mathbf{n} = -i$. $\mathbf{F}$ on S_2 is given by $(0, y^2, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore, $\int\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} \mathbf{F} \cdot \mathbf{n}\ dS = 0$.

Left Face, S_3: S_3 is given by $(x, -1, z)$, with $0 \le x \le 3$, $1 \le z \le 2$ and $\mathbf{n} = -j$. $\mathbf{F}$ on S_3 is given by $(-x^2, 1, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = -1$. Therefore,

$$\begin{aligned}\int\int_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_3} \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_3} -1\ dS \\ &= -1 \cdot \text{area}(S_3) \\ &= -1 \cdot 3 = -3.\end{aligned}$$

Right Face, S_4: S_4 is given by $(x, 2, z)$, with $0 \le x \le 3$, $1 \le z \le 2$ and $\mathbf{n} = j$. $\mathbf{F}$ on S_4 is given by $(-x^2, 4, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = 4$. Therefore,

$$\begin{aligned}\int\int_{S_4} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_4} \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_4} 4\ dS \\ &= 4 \cdot \text{area}(S_3) \\ &= 4 \cdot 3 = 12.\end{aligned}$$

Top Face, S_5: S_5 is given by $(x, y, 2)$, with $0 \le x \le 3$, $-1 \le y \le 2$ and $\mathbf{n} = k$. $\mathbf{F}$ on S_5 is given by $(-x^2, y^2, -4)$ and thus $\mathbf{F} \cdot \mathbf{n} = -4$. Therefore,

$$\begin{aligned}\int\int_{S_5} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_5} \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_5} -4\ dS \\ &= -4 \cdot \text{area}(S_5) \\ &= -4 \cdot 9 = -36.\end{aligned}$$

Bottom Face, S_6: S_6 is given by $(x, y, 1)$, with $0 \le x \le 3$, $-1 \le y \le 2$ and $\mathbf{n} = -k$. $\mathbf{F}$ on S_6 is given by $(-x^2, y^2, -1)$ and thus $\mathbf{F} \cdot \mathbf{n} = 1$. Therefore,

$$\begin{aligned}\int\int_{S_6} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_6} \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_6} 1\ dS \\ &= 1 \cdot \text{area}(S_6) \\ &= 1 \cdot 9 = 9.\end{aligned}$$

Putting these all together we have $\int\int_S \mathbf{F} \cdot d\mathbf{S} = -27 + 0 - 3 + 12 - 36 + 9 = -45$.

13. Let $\mathbf{F} = (z^2, x^2, -y^2)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path traversing counterclockwise the square with sides of length s centered at $(x_0, y_0, 0)$. Then divide this number by the area of the square and take the limit as $s \to 0$. Compare this with $(\text{Curl } \mathbf{F})(x_0, y_0, 0) \cdot k$.

Solution. In this problem we are using the limit definition to calculate $(\text{Curl } \mathbf{F})(x_0, y_0, 0) \cdot k$. For this, we must compute a line integral of $\mathbf{F}$ over each side of the square C.

Bottom side, C_1: C_1 is given by $\mathbf{r}(t) = (x_0 - \frac{s}{2}, y_0 - \frac{s}{2}, 0) + t(s, 0, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (s, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 - \frac{s}{2})^2, -(y_0 - \frac{s}{2})^2)$.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 0\ dt \\ &= 0.\end{aligned}$$

Top side, C_2: C_2 is given by $\mathbf{r}(t) = (x_0 + \frac{s}{2}, y_0 + \frac{s}{2}, 0) + t(-s, 0, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (-s, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 + \frac{s}{2})^2, -(y_0 + \frac{s}{2})^2)$.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 0\ dt \\ &= 0.\end{aligned}$$

Right side, C_3: C_3 is given by $\mathbf{r}(t) = (x_0 + \frac{s}{2}, y_0 - \frac{s}{2}, 0) + t(0, s, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (0, s, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 + \frac{s}{2})^2, -(y_0 - \frac{s}{2})^2)$.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 (x_0 + \frac{s}{2})^2 s\ dt \\ &= x_0^2 s + x_0 s^2 + \frac{s^3}{4}.\end{aligned}$$

Left side, C_4: C_4 is given by $\mathbf{r}(t) = (x_0 - \frac{s}{2}, y_0 + \frac{s}{2}, 0) + t(0, -s, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (0, -s, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 - \frac{s}{2})^2, -(y_0 + \frac{s}{2})^2)$.

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 -(x_0 - \frac{s}{2})^2 s\ dt \\ &= -x_0^2 s + x_0 s^2 + -\frac{s^3}{4}.\end{aligned}$$

We now have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + (x_0^2 + x_0 s^2 + \frac{s^3}{4}) + (-x_0^2 + x_0 s^2 + -\frac{s^3}{4}) = 2s^2 x_0.$$

Therefore,

$$\lim_{s \to 0} \frac{1}{\text{area}(S)} \int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{s \to 0} \frac{1}{s^2} 2s^2 x_0 = 2x_0.$$

To Check:

$$\text{Curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & -y^2 \end{vmatrix} = (-2y, 2z, 2x).$$

Thus, Curl $\mathbf{F}(x_0, y_0, z_0) = (-2y_0, 2z_0, 2x_0)$ and Curl $\mathbf{F}(x_0, y_0, z_0) \cdot k = 2x_0$.

14. Let C be the curve obtained by intersecting the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$, and $\mathbf{F} = -y^3 \vec{i} + x^3 \vec{j} + -z^3 \vec{k}$. Set up the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a single integral over an interval of the form $[a, b]$. Now evaluate this line integral by using Stoke's Theorem.

Solution. The curve C lies on the plane $z = 1-x-y$ but also on the cylinder $x^2+y^2 = 1$. So a parametrization and tangent of C are

$$\mathbf{r}(t) = (\cos(t), \sin(t), 1 - \cos(t) - \sin(t)) \text{ and } \mathbf{r}'(t) = (-\sin(t), \cos(t), \sin(t) - \cos(t)).$$

$\mathbf{F}(\mathbf{r}(t)) = (-\sin^3(t), \cos^3(t), -(1 - \cos(t) - \sin(t))^3)$.

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \sin^4(t) + \cos^4(t) - (1 - \cos(t) - \sin(t))^3 \cdot (\sin(t) - \cos(t)).$$

Integrating this last expression from 0 to 2π is doable .... but not much fun.

To apply Stoke's Theorem we will integrate $\nabla \times \mathbf{F}$ over S, that portion of the given plane lying above the disk $D : 0 \le x^2 + y^2 \le 1$ in the xy-plane.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (3x^2 + 3y^2)\vec{k} = (0, 0, 3x^2 + 3y^2).$$

S is given by $G(u, v) = (u, v, 1 - u - v)$, with $0 \le u^2 + v^2 \le 1$. Thus,

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1).$$

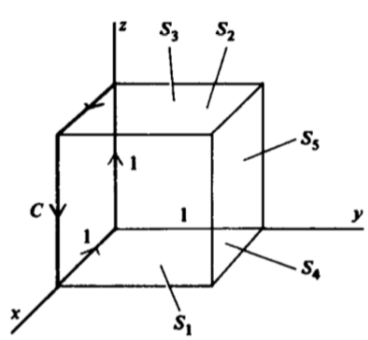
Moreover, $\nabla \times \mathbf{F}$ on S is $(0, 0, 3u^2 + 3v^2)$. Thus, on S,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{T}_u \times \mathbf{T}_v = (0, 0, 3u^2 + 3v^2) \cdot (1, 1, 1) = 3u^2 + 3v^2.$$

Therefore:

$$\begin{aligned} \int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int\int_D 3u^2 + 3v^2 \; dA \\ &= 3 \int_0^{2\pi} \int_0^1 r^2 \cdot r \; dr d\theta \\ &= 6\pi \int_0^1 r^3 \; dr \\ &= \frac{6\pi}{4} = \frac{3\pi}{2}. \end{aligned}$$

15. Verify Stoke's Theorem for $\mathbf{F} = (z^2, -y^2, 0)$ and C the square of side 1 oriented as shown, lying in the xz-plane and S the open box with sides S_1, S_2, S_3, S_4, S_5. What happens, if instead, you take S to be the square enclosed by C?



Solution. Both terms in Stoke's Theorem require computing several integrals. We start by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C = C_1 \cup C_2 \cup C_3 \cup C_4$ is the curve indicated in the diagram.

$C_1 : \mathbf{r}(t) = (1-t, 0, 0),\ 0 \le t \le 1,\ \mathbf{r}'(t) = (-1, 0, 0),\ \mathbf{F}(\mathbf{r}(t)) = (0, 0, 0).$

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 0\ dt \\ &= 0.\end{aligned}$$

$C_2 : \mathbf{r}(t) = (0, 0, t),\ 0 \le t \le 1,\ \mathbf{r}'(t) = (0, 0, 1),\ \mathbf{F}(\mathbf{r}(t)) = (t^2, 0, 0).$

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 0dt \\ &= 0.\end{aligned}$$

$C_3 : \mathbf{r}(t) = (t, 0, 1),\ 0 \le t \le 1,\ \mathbf{r}'(t) = (1, 0, 0),\ \mathbf{F}(\mathbf{r}(t)) = (1, 0, 0).$

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 1\ dt \\ &= 1.\end{aligned}$$

$C_4 : \mathbf{r}(t) = (1, 0, 1-t),\ 0 \le t \le 1,\ \mathbf{r}'(t) = (0, 0, -1),\ \mathbf{F}(\mathbf{r}(t)) = ((1-t)^2, 0, 0).$

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 0\ dt \\ &= 0.\end{aligned}$$

We now have,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 1 + 0 = 1.$$

To calculate the curl of $\mathbf{F}$

$$\text{Curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -y^2 & 0 \end{vmatrix} = (0, 2z, 0).$$

For the surface integral $\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S}$ is the sum of the surface integrals of the curl of $\mathbf{F}$ over the five faces indicated in the diagram.

Front Face, S_1: S_1 is given by $(1, y, z)$, with $0 \le y \le 1$, $0 \le z \le 1$ and $\mathbf{n} = i$. Curl $\mathbf{F}$ on S_1 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$\begin{aligned}\int\int_{S_1} \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_1} \text{Curl } \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_1} 0\ dS \\ &= 0.\end{aligned}$$

Back Face, S_2: S_2 is given by $(0, y, z)$, with $0 \le y \le 1$, $0 \le z \le 1$ and $\mathbf{n} = -i$. Curl $\mathbf{F}$ on S_2 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$\begin{aligned}\int\int_{S_2} \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_2} \text{Curl } \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_3} 0\ dS \\ &= 0.\end{aligned}$$

Top Face, S_3: S_3 is given by $(x, y, 1)$, with $0 \le x \le 1$, $0 \le y \le 1$ and $\mathbf{n} = k$. Curl $\mathbf{F}$ on S_3 is given by $(0, 2, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$\begin{aligned}\int\int_{S_3} \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_3} \text{Curl } \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_3} 0\ dS \\ &= 0.\end{aligned}$$

Bottom Face, S_4: S_4 is given by $(x, y, 0)$, with $0 \le x \le 1$, $0 \le y \le 1$ and $\mathbf{n} = -k$. Curl $\mathbf{F}$ on S_4 is given by $(0, 0, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$\begin{aligned}\int\int_{S_4} \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_4} \text{Curl } \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_4} 0\ dS \\ &= 0.\end{aligned}$$

Right Face, S_5: S_5 is given by $(x, 1, z)$, with $0 \le x \le 1$, $0 \le z \le 1$ and $\mathbf{n} = j$. Curl $\mathbf{F}$ on S_5 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 2z$. Therefore,

$$\begin{aligned}\int\int_{S_5} \text{Curl } \mathbf{F} \cdot d\mathbf{S} &= \int\int_{S_5} \text{Curl } \mathbf{F} \cdot \mathbf{n}\ dS \\ &= \int\int_{S_5} 2z\ dS \\ &= \int_0^1 \int_0^1 2z\ dz\ dx \\ &= \int_0^1 2z\ dz \\ &= 1.\end{aligned}$$

Putting these all together we have $\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + 0 + 0 + 1 = 1$, as expected, thereby confirming Stoke's Theorem.

Finally, an important consequence of Stoke's theorem is that the surface integrals of Curl $\mathbf{F}$ over two surfaces sharing a common oriented boundary are the same. Let S_0 be the left face of the square in the diagram, so that S and S_0 share the same oriented boundary.

S_0 is given by $(x, 0, z)$, with $0 \le x \le 1$, $0 \le z \le 1$ and $\mathbf{n} = j$. Note that j is the correct normal when considering S_0 as an open, oriented surface with boundary C. If we were considering S_0 as the 6th side of the cube, we would take $-j$ as the unit normal. Curl $\mathbf{F}$ on S_0 is given by $(0, 2z, 0)$ and thus Curl $\mathbf{F} \cdot \mathbf{n} = 2z$.

Therefore,

$$\begin{aligned}\int\int_{S_0} \text{Curl } \mathbf{F}\cdot d\mathbf{S} &= \int\int_{S_0} \mathbf{F}\cdot\mathbf{n}\ dS\\ &= \int\int_{S_0} 2z\ dS\\ &= \int_0^1\int_0^1 2z\ dz\ dx\\ &= \int_0^1 2z\ dz\\ &= 1\\ &= \int\int_S \text{Curl } \mathbf{F}\cdot d\mathbf{S}.\end{aligned}$$

16. Calculate, without using Stoke's Theorem, $\int\int_{S_1} \nabla\times\mathbf{F}\cdot d\mathbf{S}$, for $\mathbf{F} = (3y^2+2y)\vec{i} + 3z^2\vec{j} + 3x^2\vec{k}$ and S_1 the inverted cone $z = 1-\sqrt{x^2+y^2}$, with vertex $(0,0,1)$, and $z \geq 0$. Then calculate directly $\int\int_{S_2} \nabla\times\mathbf{F}\cdot d\mathbf{S}$, for S_2 the unit disk in the xy-plane. The answers you get should be the same. This shows the consequence of Stoke's Theorem, that surfaces integrals of the curl of a vector field over surfaces sharing the same boundary are independent of the surface.

Solution. $\nabla\times\mathbf{F} = \begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3y^2+2y & 3z^2 & 3x^2\end{vmatrix} = (-6z, -6x, -6y-2)$.

If we integrate directly over the cone, we use the parametrization

$$G(u,v) = (v\cos(u), v\sin(u), 1-v),$$

with $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$. It follows that

$$\nabla\times\mathbf{F}(G(u,v)) = (-6(1-v), -6v\cos(u), -6v\sin(u)-2).$$

Moreover,

$$\mathbf{T}_u\times\mathbf{T}_v = (v\cos(u), v\sin(u), -v).$$

Note that this vector, has a negative z-component, and thus is *inside* of the inverted cone. If we flatten the cone, this would point downward, and contradict the right hand thumb rule. Thus, we need to use the vector $-(\mathbf{T}_u\times\mathbf{T}_v)$ when integrating $\nabla\mathbf{F}$. (An important point: We could also parametrize the inverted cone using $H(u,v) = (u, v, 1-\sqrt{u^2+v^2})$, with $0 \leq u^2+v^2 \leq 1$, and in this case $\mathbf{T}_u\times\mathbf{T}_v$ gives the correct normal vector.)

We now have

$$\{\nabla\times\mathbf{F}(G(u,v))\}\cdot -(\mathbf{T}_u\times\mathbf{T}_v) = 6v(1-v)\cos(u) + 6v^2\sin(u)\cos(u) - 6v^2\sin(u) - 2v.$$

When we calculate $\int\int_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}$, first integrating with respect to u, the trig terms integrated from 0 to 2π all become 0. We are left with,

$$\begin{aligned}\int_0^1\int_0^{2\pi} -2v\ dudv &= 2\pi\int_0^1 -2v\ dv\\ &= -4\pi\cdot\frac{v^2}{2}\Big|_0^1\\ &= -2\pi.\end{aligned}$$

To integrate over the disk we have $G(u,v) = (u,v,0)$ with $0 \leq u^2+v^2 \leq 1$ and $\mathbf{T}_u\times\mathbf{T}_v = (0,0,1)$. $\nabla\times\mathbf{F}(G(u,v)) = (0, -6u, -6v-2)$, so $\nabla\times\mathbf{F}(G(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v = -6v-2$. Thus,

$$\begin{aligned}
\int\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int\int_{S_2} -6v - 2 \ dA \\
&= \int_0^{2\pi} \int_0^1 (-6r\sin(\theta) - 2) \ rdrd\theta \\
&= \int_0^{2\pi} \int_0^1 -6r^2\sin(\theta) - 2r \ dr \ d\theta \\
&= \int_0^{2\pi} -2\sin(\theta) - 1 \ dr \\
&= -2\pi,
\end{aligned}$$

which is what we want.